Symmetric tensors with only real eigenvectors

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Motivation: rank one approximation of a tensor

A real n^d-tensor $\mathcal{A} \in \underbrace{\mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n}_{d}$ is said to be of **rank one** if $\mathcal{A} = x_1 \otimes \cdots \otimes x_d$ for some $x_1, \ldots, x_n \in \mathbb{R}^n$.
Denote by X the set of real n^d-tensors of rank one and for a general n^d-tensor $\mathcal{A}$ define the distance function from $\mathcal{A}$ to X:

$$\mathrm{dist}_{\mathcal{A}} : X \to \mathbb{R}$$
$$x = x_1 \otimes \cdots \otimes x_n \mapsto \|\mathcal{A} - x\|^2 = (\mathcal{A}, \mathcal{A}) - 2(\mathcal{A}, x) + (x, x)$$

Goal: Find the best rank one approximation of a tensor (minimize $\mathrm{dist}_{\mathcal{A}}$).
To keep in mind: Approximate complicated tensors by simple ones.
Minimizer(s) of $\mathrm{dist}_{\mathcal{A}}$ are among its critical points.
When the tensor $\mathcal{A}$ is symmetric the critical points of $\mathrm{dist}_{\mathcal{A}}$ are $\lambda \cdot x \otimes \cdots \otimes x$, where $x \in \mathbb{R}^n$ is a (unit) **eigenvector** of $\mathcal{A}$ and $\lambda \in \mathbb{R}$ is the corresponding **eigenvalue**.

Spectral theory of matrices (Cauchy, 1829)

$\mathcal{A} = \{a_{ij}\}$ - $n \times n$ matrix. Non-zero vectors $x \in \mathbb{C}^n$ satisfying

$$\operatorname{rank} \begin{pmatrix} \sum_{j=1}^{n} a_{1j}x^j & \dots & \sum_{j=1}^{n} a_{nj}x^j \\ x^1 & \dots & x^n \end{pmatrix} = 1$$

are **eigenvectors** of $\mathcal{A}$. (make sure it is just $\mathcal{A}x = \lambda x$ for some $\lambda \in \mathbb{C}$)
A **generic** $n \times n$ matrix defines n complex **eigenpoints** $x \in \mathbb{C}P^{n-1}$.
If $\mathcal{A}$ is symmetric ($a_{ij} = a_{ji}$) all its eigenpoints are real ($x \in \mathbb{R}P^{n-1}$).
For a symmetric matrix $\mathcal{A}$ consider its quadratic form $f_{\mathcal{A}}(x) = x^t\mathcal{A}x$.
Critical points of $f_{\mathcal{A}} : S^{n-1} \to \mathbb{R}$ are eigenvectors of $\mathcal{A}$:

$$\nabla f_{\mathcal{A}}(x) = \lambda \nabla |x|^2, \quad \lambda \in \mathbb{R},\, x \in S^{n-1} \Longleftrightarrow \mathcal{A}x = \lambda x$$

Moreover, $f_{\mathcal{A}}$ is a **Morse** function iff $\mathcal{A}$ is generic.

Spectral theory of tensors (Lim and Qi, 2005)

$\mathcal{A} = \{a_{i_1 \dots i_n}\}$ - real n^d-tensor (coordinate representation of $\mathcal{A} \in (\mathbb{R}^n)^{\otimes d}$).
Non-zero vectors $x \in \mathbb{C}^{n-1}$ satisfying

$$\operatorname{rank} \begin{pmatrix} \sum_{i_2,\dots,i_d=1}^{n} a_{1 i_2 \dots i_d} x^{i_2} \dots x^{i_d} & \dots & \sum_{i_2,\dots,i_d=1}^{n} a_{n i_2 \dots i_d} x^{i_2} \dots x^{i_d} \\ x^1 & \dots & x^n \end{pmatrix} = 1$$

are **eigenvectors** of $\mathcal{A}$ [4, 6].

A generic tensor defines $\frac{(d-1)^n - 1}{d-2} = \sum_{i=0}^{n-1} (d-1)^i$ eigenpoints in $\mathbb{C}P^{n-1}$ [3].

$\mathcal{A}$ is said to be **symmetric** if its entries are invariant w.r.t. permutations of their indices. Symmetric tensors correspond to homogeneous polynomials:

$$\mathcal{A} = \{a_{i_1 \dots i_d}\} \quad \Longleftrightarrow \quad f_{\mathcal{A}}(x) = \sum_{i_1,\dots,i_d=1}^{n} a_{i_1 \dots i_d} x^{i_1} \dots x^{i_d}$$

Symmetric tensors

Real eigenpoints $x \in \mathbb{R}P^{n-1}$ correspond to critical points of $f_{\mathcal{A}} : S^{n-1} \to \mathbb{R}$:

$$\nabla f_{\mathcal{A}}(x) = \lambda \nabla |x|^2, \quad \lambda \in \mathbb{R}, x \in S^{n-1}$$

"Most" tensors have less than $\frac{(d-1)^n - 1}{d-2}$ real eigenpoints [2, 5].

Question

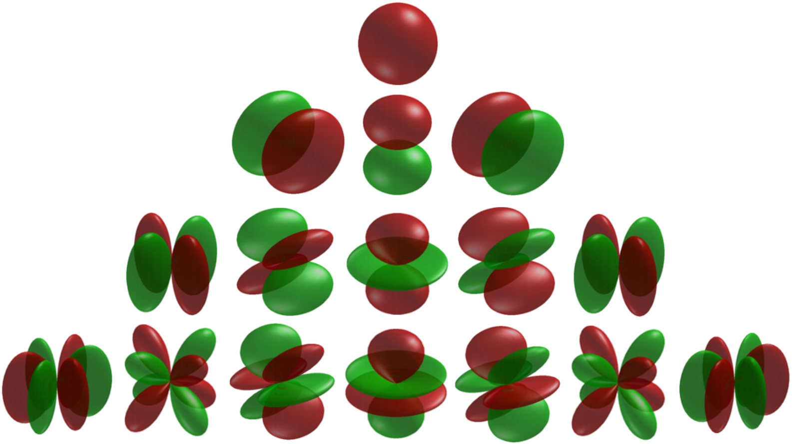
Do there exist generic tensors with only real eigenpoints?

$n = 2$: trivially yes.
$n = 3$: yes, the product of general linear forms $(v_1, x) \cdot \cdots \cdot (v_d, x), v_i \in \mathbb{R}^n$ defines such a tensor [1].
$n > 3$: was unknown (Conjecture of Abo, Seigal, Sturmfels [1], 2015).

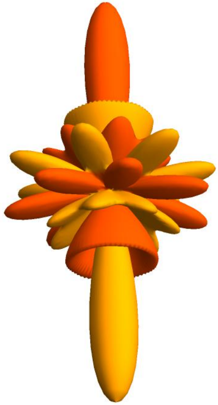
Spherical harmonics

The spherical Laplacian $-\Delta : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ has spectrum $\{d(d+n-2)\}_{d\geq 0}$ and the eigenfunctions corresponding to $d(d+n-2)$ are the restrictons to S^{n-1} of harmonic homogeneous polynomials of degree d.

Consider now $n = 3$.



$P_d(t)$ - degree d Legendre polynomial,
$Y_{d,0}(\theta,\varphi) = P_d(\cos(\theta))$ –
zonal harmonic of degree d,
$Y_{d,d}(\theta,\varphi) = \sin^d\theta \cdot \cos(d\varphi)$ –
sectorial harmonic of degree d.
Any small perturbation $Y_{d,0} + \varepsilon \cdot Y_{d,d}$
is a Morse function which has
$2[(d-1)^2 + (d-1) + 1]$ critical points on S^2.
In high dimensions $n > 3$ instead of $Y_{d,d}$ take
a Morse spherical harmonic in $n-1$ variables with
$2[(d-1)^{n-2} + \cdots + (d-1) + 1]$ critical points.



d=7

Theorem

For any $n \geq 2$ and any $d \geq 0$ there exist a spherical harmonic of degree d in n variables which defines $\frac{(d-1)^n - 1}{d-2}$ real eigenpoints.

References

[1] *H. Abo, A. Seigal and B. Sturmfels*, Eigenconfigurations of tensors, Algebraic and Geometric Methods in Discrete Mathematics, Contemporary Mathematics, vol. 685, 1–25, 2016.

[2] *P. Breiding*, The expected number of Z-eigenvalues of a real gaussian tensor, arXiv:1604.03910 [math.AG], 2016.

[3] D. Cartwright, B. Sturmfels. *The number of eigenvalues of a tensor*, Linear Algebra and its Applications, 438 (2013), 942-952.

[4] L. H. Lim. *Singular values and eigenvalues of tensors: a variational approach*, Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, 1 (2005), 129-132

[5] *M. Maccioni*, The number of real eigenvectors of a real polynomial, arXiv:1606.04737 [math.AG], 2016.

[6] L. Qi. *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Computation, 40 (2005), 1302-1324.

Thanks!