SVD approximations for large-scale inverse problems

Clarissa Garvey   Chang Meng   James Nagy

Department of Mathematics and Computer Science

EMORY UNIVERSITY

ccgarve@emory.edu

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Singular Value Decomposition (SVD)

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The columns of U and V are the left and right singular vectors of A, respectively.

The σ_i are the singular values of A.

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- $\mathbf{b} \in \mathbb{R}^N$ is the measured data,
- $\mathbf{x} \in \mathbb{R}^N$ is the desired, unknown solution,
- $K \in \mathbb{R}^{N \times N}$ is a matrix relating the true solution to observation, and
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Then direct solution using the SVD may amplify noise:

$$K^{-1}\mathbf{b} = \mathbf{x} + K^{-1}\mathbf{e} = \mathbf{x} + V\Sigma^{-1}U^T\mathbf{e}$$

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$\{\phi_i\}$ are weights such that, for small σ_i, $\phi_i = 0$.

In a truncated SVD, $\phi_i = 0$ if $i > k$, a cutoff index.

The Kronecker Product

The Kronecker product of matrices $A_{m \times m}$ and B is

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \dots & a_{1,m}B \\ \vdots & & \vdots \\ a_{m,1}B & \dots & a_{m,m}B \end{bmatrix}$$

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$$A \otimes B = (U_A \otimes U_B)(\Sigma_A \otimes \Sigma_B)(V_A \otimes V_B)^T$$

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But all $N \times N$ matrices K can be represented as

$$K = \sum_{i=1}^{R} A_i \otimes B_i$$

where all A_i, B_i are $n \times n$ with $n^2 = N$ in size and the index R is the Kronecker rank. (See *Van Loan and Pitsiannis, 1993 [**?**]*)

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Further, $K = U_1 \Sigma_1 V_1^T + \sum_{i=2}^{R} U_1 U_1^T K_i V_1 V_1^T$

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\begin{aligned}
\text{Further, } K &= U_1 \Sigma_1 V_1^T + \sum_{i=2}^{R} U_1 U_1^T K_i V_1 V_1^T \\
&= U_1 \left(\Sigma_1 + \sum_{i=2}^{R} U_1^T K_i V_1 \right) V_1^T
\end{aligned}
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See *Nagy, Ng, and Perrone (2006) [**?**]* and *Kamm and Nagy (2000) [**?**]*

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$$= \widehat{U} \left(\begin{bmatrix} \widehat{\Sigma}_k & 0 \\ 0 & \widehat{\Sigma}_0 \end{bmatrix} + \begin{bmatrix} 0 & \widehat{W}_{12} \\ \widehat{W}_{21} & W_{22} \end{bmatrix} \right) \widehat{V}^T.$$

Algorithm

Given

$$K = \widehat{U}\left(\begin{bmatrix}\widehat{\Sigma}_k & 0 \\ 0 & \widehat{\Sigma}_0\end{bmatrix} + \begin{bmatrix}0 & \widehat{W_{12}} \\ \widehat{W_{21}} & W_{22}\end{bmatrix}\right)\widehat{V}^T$$

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partition $\widehat{U} = \begin{bmatrix} \widehat{U}_k & \widehat{U}_0 \end{bmatrix}$ and $V = \begin{bmatrix} \widehat{V}_k & \widehat{V}_0 \end{bmatrix}$.

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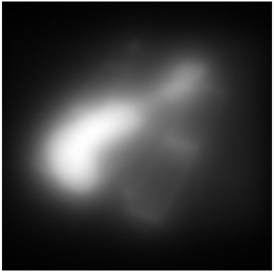
Then we can construct a truncated SVD

$$K \approx \widehat{U}_k \widehat{\Sigma}_k \widehat{V}_k^T$$

Image Deblurring Example



observed image

SVD approx
1 Kron term

SVD approx
5 Kron terms

In this example:

- K is $N \times N$, with $N = 65,536 \times 65,536$.
- K_{TSVD} uses $k = 2,601$ for truncation.

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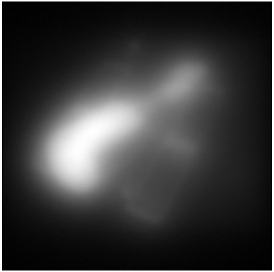

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SVD approx
1 Kron term

SVD approx
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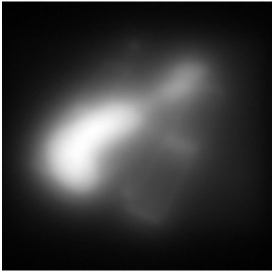

observed image

SVD approx

1 Kron term

SVD approx

67 Kron terms

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Conclusion

- We created an algorithm for approximating a truncated SVD.
- Kronecker products help keep computational costs down.
- We can compute the TSVD on larger problems than traditional methods allow.
- This method is fast (seconds) and produces reasonable solutions.

References and Acknowledgement

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Thank you!