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Nahuel Foresta

Department of Mathematics
Politecnico di Milano

# Optimal Switching for Marked Point Processes

A Day in Applied Mathematics

Milan - September 12, 2017

# Outline

- Introduction to Optimal Switching
- Marked Point Processes
- Optimal Switching for MPP
- System of Reflected BSDE and Optimal Switching

## Optimal switching in general

- We have a stochastic system that can evolve with $m$ different dynamics, $X^i$ for $i = 1, \ldots, m$.
- To each dynamic may also correspond different running rewards $f^i$ and final rewards $\xi^i$.
- A controller can decide at any time $s$ to switch from mode $i$ to $j$, but with a cost $C_s(i, j)$

A strategy for the controller is a sequence of random switch times and switch actions $\mathbf{a} = (\theta_k, \alpha_k)$. Using this strategy we obtain a switched dynamic $X^{\mathbf{a}}$.

Let $a_s$ be the process specifying which mode is active at time $s$ for a given strategy.

For a given strategy $\mathbf{a}$ we obtain the reward:

$$\mathbb{E}\left[\int_0^T f_s^{a_s}(X_s^{\mathbf{a}})ds + g(a_T, X_T^{\mathbf{a}})\right]$$

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The value of the problem if we start at time $t$ in mode $i$ is

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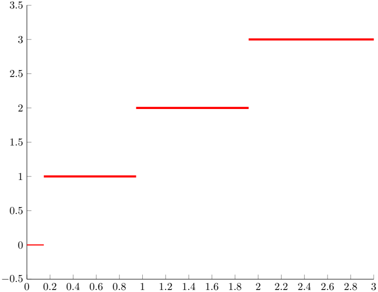


Figure: An example of univariate point process

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space, $(E, \mathcal{E})$ a measurable space, $T$ a fixed time horizon. A point process is a sequence of r.v. $(T_n, \xi_n)$ on $R_+ \times E$ such that:

$$T_n \leq T_{n+1}$$

$$T_n < T_{n+1} \text{ if } T_n \leq T$$

We can associate an integer random measure

$$p(dsde) = \sum_{n \geq 1} \delta_{T_n, \xi_n}(dsde)$$

that counts the number of jumps that happen in the subset *dsde*.
Let $\mathcal{F}_t$ be the completed filtration generated by $p$.

There exist $\phi$ and $A$ such that

$$q(\omega, dsde) = p(\omega, dsde) - \phi_s(\omega, de)dA_s(\omega)$$

is a martingale measure, that is for all $\mathcal{P} \otimes \mathcal{E}$-measurable process $H$ such that $\mathbb{E}\left[\int_0^T \int_E |H_s|(e)\phi_s(de)dA_s\right] < \infty$,

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For $i = 1, \ldots, M$ modes the following are given

- A terminal gain $\xi^i$
- A running gain $f_t^i$
- The costs $C_t(i,j)$ of switching from mode $i$ to mode $j$ at time $t$
- $\rho^i$ predictable fields on $\Omega \times [0,T] \times E$, which will be used to determine the law of the process.

The controller chooses a strategy $\mathbf{a} = (\theta_k, \alpha_k)$. Define

$$a_t = \sum_{k \geq 1} \alpha_{k-1} \mathbb{1}_{(\theta_{k-1}, \theta_k]}(t)$$

the indicator of the current strategy.

For each $\mathbf{a} \in \mathcal{A}_t^i$ define

$$\rho_t^{\mathbf{a}}(e) = \sum_{k \geq 1} \mathbb{1}_{(\theta_k, \theta_{k-1}]}(t) \rho_t^{\alpha_k}(e)$$

For each $\mathbf{a}$ $\rho^{\mathbf{a}}$ introduces a new probability $\mathbb{P}^{\mathbf{a}} \ll \mathbb{P}$ under which the compensator is $\rho^{\mathbf{a}}(e)\phi_s(de)dA_s$.

The value function of the problem is then

$$v(t,i) = \operatorname*{ess\,sup}_{\mathbf{a} \in \mathcal{A}_t^i} \mathbb{E}^{\mathbf{a}} \left[ \xi^{a_T} + \int_t^T f_s^{a_s} dA_s - \sum_{k \geq 1} C_{\theta_k}(\alpha_{k-1}, \alpha_k) \middle| \mathcal{F}_t \right]$$

$\mathcal{A}_t^i$ denotes the set of strategies starting from time $t$ in mode $i$.

In [Hu and Tang, 2010], [Hamadène and Zhang, 2010]... a system of interconnected BSDEs is used to solve switching problems.

In our case:

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_s^i dA_s + \int_t^T \int_E U_s^i(e)(\rho_s^i(e) - 1)\phi_s(de)dA_s \\ \qquad\qquad - \int_t^T \int_E U_s^i(e) q(dsde) + K_T^i - K_t^i \\ Y_t^i \geq \max_{j \in \mathbb{A}_i} (Y_t^j - C_t(i,j)) \\ \int_0^T (Y_t^i - \max_{j \in \mathbb{A}_i} (Y_t^j - C_t(i,j))) dK_t^i = 0. \end{cases}$$

$Y^i$ càdlàg, $U^i$ predictable, $K^i$ continuous increasing such that

$$\mathbb{E}\left[\int_0^T \int_E e^{\beta A_s}(|Y_s^i|^2 + |U_s^i(e)|^2)\phi_s(de)dA_s\right] + \mathbb{E}\left[(K_T^i)^2\right] < \infty$$

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- There exists $L$ such that $0 \le \rho_t^i(e) \le L$
- There exists $\gamma > 3 + L^4$ such that $\mathbb{E}e^{\gamma A_T} < \infty$.

For some $\hat{\beta} > L$:

- $\xi^i$ $\mathcal{F}_T$-measurable such that $\mathbb{E}[e^{\hat{\beta} A_T}|\xi^i|^2] < \infty$
- $f_s^i$ is a progressive process such that

$$\mathbb{E}\left[\int_0^T e^{\hat{\beta} A_s}|f_s^i|^2 dA_s^i\right] < \infty.$$

- $C_t(i,j)$ for $i,j \in \mathcal{J}$ adapted continuous. $C_t(i,j) > 0$ for $i \neq j$, $C_t(i,i) = 0$ and

$$\inf_t (C_t(i,j) + C_t(j,l) - C_t(i,l) > 0 \text{ for all } i \neq j \neq l. \quad (1)$$

for all $i,j \in \mathcal{J}$, $\mathbb{E}\left[\sup_t e^{\hat{\beta} A_t} C_t(i,j)\right] < \infty$.

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**Theorem**

*Under the previous assumptions, there exists a solution to the system of RBSDE with $(Y^i, U^i) \in \mathbb{H}^\beta$ with $L < \beta < \hat{\beta}$.*

## Theorem

*Let $(Y^i, U^i, K^i)_{i\in\mathcal{J}}$ be a solution to the system of RBSDE. Then $Y^i_t = v(t,i)$, i.e.*

$$Y_t = \operatorname*{ess\,sup}_{\mathbf{a}\in\mathcal{A}^i_t} \mathbb{E}^{\mathbf{a}} \left[ \xi^{a_T} + \int_t^T f_s^{a_s} dA_s - \sum_{k\geq 1} C_{\theta_k}(\alpha_{k-1}, \alpha_k) \middle| \mathcal{F}_t \right]$$

*The sequence defined as $(\theta^*_0, \alpha^*_0) = (t,i)$ and*

$$\theta^*_n = \inf\left\{ s \geq \theta^*_{n-1} : Y_s^{\alpha^*_{n-1}} = \max_{j\in\mathbb{A}_{\alpha^*_{n-1}}} (Y^j_s - C_s(\alpha^*_{n-1}, j)) \right\}$$

$$\alpha^*_n = \arg\max_{j\in\mathbb{A}_{\alpha^*_{n-1}}} \left( Y^k_{\theta^*_n} - C_{\theta^*_n}(\alpha^*_{n-1}, j) \right)$$

*is an optimal strategy.*

# Thank you for your attention

Nahuel Foresta nahueltomas.foresta@polimi.it